UNLIMITED
UNCLASSIFIED

ILLIMITÉ
NON CLASSIFIÉ

8

Division of Mechanical
Engineering Report

Rapport de la Division
de génie mécanique

1983/11

DM-3
NRC NO. 22797

EQUATIONS OF MOTION FOR RIGID MULTIBODY SYSTEMS

I.H. Mufti

Division of
Mechanical Engineering

Division de
génie mécanique

National Research Council Canada
Conseil national de recherches Canada

Canada

DIVISION OF MECHANICAL ENGINEERING PUBLICATIONS

DM **(Division of Mechanical Engineering Report)**
Scientific and technical information considered important, complete and a lasting contribution to existing knowledge.

TR **(Technical Report)**
Information less broad in scope but a substantial contribution to existing knowledge.

CTR **(Controlled/Classified Technical Report)**
A Technical Report with controlled distribution for national security, proprietary or other reasons.

LM **(Laboratory Memorandum)**
Preliminary or exploratory information with controlled distribution.

CAT **(Calibration Analysis and Test Report)**
Information on minor laboratory projects or services.

PUBLICATIONS DE LA DIVISION DE GÉNIE MÉCANIQUE

DM **(Rapport de la Division de génie mécanique)**
Informations scientifiques et techniques jugées importantes, complètes et susceptibles de contribuer de façon durable à l'avancement des connaissances courantes.

TR **(Rapport technique)**
Informations de moindre importance, mais pouvant contribuer substantiellement à l'avancement des connaissances actuelles.

CTR **(Rapport technique à diffusion contrôlée/classifiée)**
Rapport technique à diffusion contrôlée pour des raisons de sécurité nationale, de propriété intellectuelle et autres.

LM **(Mémoire de laboratoire)**
Informations préliminaires ou de nature exploratoire à diffusion contrôlée.

CAT **(Rapport d'étalonnage d'analyse et d'essai)**
Informations sur de petits projets ou des services de laboratoire.

UNLIMITED
UNCLASSIFIED

ILLIMITÉ
NON CLASSIFIÉ

EQUATIONS OF MOTION FOR RIGID MULTIBODY SYSTEMS

ÉQUATIONS DU MOUVEMENT DE SYSTÈMES À PLUSIEURS CORPS SOLIDES

I.H. Mufti

Division of mechanical
Engineering Report

Rapport de la Division
de génie mécanique

1983/11

DM-3
NRC No. 22797

P.A. Hamil, Head/Chef
Systems Laboratory/
Laboratoire des systèmes

E.H. Dudgeon
Director/Directeur

RESUME

Several computer-oriented methods for the formulation of the equations of motion for multibody systems with tree structure and also with closed chains are presented in a unified manner. They all stem from the same basic set of equations which are regarded as arising out of the stationary point condition of a quadratic programming (QP) problem. Using QP theory it is shown that the various methods are just different ways of solving the Lagrangian system of equations.

RÉSUMÉ

On présente dans un format homogène plusieurs méthodes informatisées pour l'établissement des équations du mouvement de systèmes à plusieurs corps avec structure en arbre et avec chaînes fermées. Ces méthodes découlent toutes du même ensemble d'équations de base qui représentent la condition de point stationnaire d'un problème de programmation quadratique (PQ). À l'aide de la théorie de la PQ, on montre que les différentes méthodes sont simplement des façons différentes de résoudre le système d'équations de Lagrange.

CONTENTS

		Page
	ABSTRACT	(iii)
	ILLUSTRATIONS	(iv)
	APPENDICES	26
	SYMBOLS	(v)
1.	INTRODUCTION	1
2.	DESCRIPTION OF THE INTERCONNECTIONS	2
3.	EQUATIONS OF MOTION FOR SYSTEMS WITH TREE STRUCTURE	4
	3.1 Systems with Ball-and-Socket Joints	6
	3.2 Systems With Ball-and-Socket Joints, Universal Joints and Pin Joints	16
4.	EQUATIONS OF MOTION FOR SYSTEMS WITH CLOSED CHAINS	21
5.	CONCLUSIONS	24
6.	REFERENCES	24
	DOCUMENTARY PAGE	

ILLUSTRATIONS

Figure		
1a	A 6-body system with tree structure	2
1b	System graph of a 6-body system	2
2	Directed tree representation of the 6-body system	2
3a	Regular labeling	3
3b	An arbitrary labeling	3
4	Free-body diagram of body i	4
5	Vectors depicting the constraint at hinge j	7
6	A 6-body system with closed chains	21

SYMBOLS

Symbol	Definition
b_i	Body i (i=1,2,...n)
c_{ij}	Elements of the incidence matrix (i=1,2,...,n;j=1,2,...,n-1)
C	Incidence matrix
C_c	Incidence matrix for closed chains
C^*	Incidence matrix due to cut hinges
C_i	Centre of mass of body i (i=1,2,...n)
d_{ii}	Vector from the barycenter of b_i to C_i(i=1,2,...,n)
d_{ik}	Vector from the barycenter of b_i to hinge point j, for each body k connected to b_i directly or indirectly at hinge j (i=1,2,...,n; k=1,2,...n; i≠ k)
D	Matrix with elements d_{ik} (i=1,2,...,n, k=1,2,...,n)
E_n	nxn unit matrix
e_1	First column of matrix E_n
f_j	Force at hinge point j (j=1,2,...,n-1)
f	Force vector with components f_j
F_i	Resultant external force on body i, acting at C_i(i=1,2,...,n)
F	Force vector with components F_i
g_j	Torque at hinge point j (j=1,2...,n-1)
g	Torque vector with components g_j
g^c	Constraint torque vector

G_i	Resultant external torque on body i (i=1,2,...,n)
G^i	Torque vector with components G_i
h_i	Angular momentum of body i(i=1,2,...,n)
h	Angular momentum vector with components h_i
h_j	Arc j or joint j (j=1,2,...,n-1)
I_i	Inertia matrix of body i about its centre of mass (i=1,2,...,n)
I	Diagonal matrix with elements $R_i I_i R_i^T$
K	System inertia matrix
ℓ_{ij}	Vector from C_i to hinge point j (i=1,2,...,n; j=1,2,...,n-1)
L	Matrix with elments $c_{ij} R_i \ell_{ij}$
m_i	Mass of body i (i=1,2,...,n)
m	Total mass of the system
M	Diagonal matrix with elements m_i or $m_i E_3$
n	Number of rigid bodies in the system
n_j	Number of rotational degrees of freedom at joint j(j=1,2,...,n-1)
n_j'	$n_j' = 3-n_j$
N	Matrix of nonlinear elements (see Eq. (38))
O	Origin of an inertial frame
O	Zero matrix, zero vector
p_{jk}	Unit vector along the axis of rotation at joint j(j=1,2,...n-1); k=1,...,n_j)
$\bar{p}_j$	Unit vector in the direction of translational motion at joint j
P_j	Matrix with elements p_{jk} (j=1,2,...,n-1)
P	Diagonal matrix with elements P_j
q_{jk}	Unit vector in the constraint direction (j=1,2,...,n-1; k=1,...n_j')
$\bar{q}_{j1}, \bar{q}_{j2}$	Unit vectors perpindicular to $\bar{p}_j$
Q_j	Matrix with elements q_{jk} (j=1,2,...,n-1)
Q	Diagonal matrix with elements Q_j

r_i	Position vector of C_i from 0 (i=1,2,...,n)
$r,\dot{r},\ddot{r}$	Vector with components r_i, $\dot{r}_i$ and $\ddot{r}_i$ respectively
R_i	Transformation matrix connecting the ith frame to the inertial frame (i=1,2,...,n)
S	Matrix defined by $(-e_1\ C)$
T	Transpose of a matrix
T	Inverse of matrix S
T_1	First row of matrix T
T_2	Last (n-1) rows of matrix T
u_1,u_2,u_3	Components of force defined in Eq. (10)
1_n	An n-vector with unit elements
$\gamma_{jk},\dot{\gamma}_{jk},\ddot{\gamma}_{jk}$	Relative angular position, velocity and acceleration respectively, at joint j(j=1,2,...,n-1; k=1,...,n_j)
$\gamma_j,\dot{\gamma}_j,\ddot{\gamma}_j$	Vectors with components γ_{jk}, $\dot{\gamma}_{jk}$ and $\ddot{\gamma}_{jk}$ respectively (j=1,2,...,n-1)
$\gamma,\dot{\gamma},\ddot{\gamma}$	Vectors with compoentns γ_j, $\dot{\gamma}_j$, and $\ddot{\gamma}_j$ respectively
δ_{ij}	Kronecker delta
λ	Vector Lagrange multiplier (see Eq. (49)
λ_c	Vector Lagrange multiplier (see Eq. (63)
μ	Matrix defined in Eq. (26)
ω_i	Absolute angular velocity of body i (i=1,2...,n)
$\dot{\omega}_i$	Absolute angular acceleration of body i (i=1,2,...,n)
$\dot{\omega}$	Vector with components $R_i\dot{\omega}_i$
Ω_j	Relative angular velocity at joint j (j=1,2,...n-1)
Ω	Vector with components Ω_j
.	First derivative with respect to time variable t
..	Second derivative with respect to time variable t

SYMBOLS (Cont')

$\triangleq$	Equals by definition
$j+, j-$	Initial vertex and final vertex of arc h_j respectively
$\times$	Cross product
$\tilde{\ }$	Used to define the cross product (see Eq. (11)
$(\ )^{-1}$	Inverse of a matrix
diag	Diagonal matrix

EQUATIONS OF MOTION FOR RIGID MULTIBODY SYSTEMS

1. INTRODUCTION

The dynamic behaviour of manipulators, linkages in machines and human body mechanism can be studied by utilizing the equations of motion derived from rigid body dynamics. The general principles involved in formulating these equations have been known since the days of Euler (1707-1783), d'Alembert (1717-1783) and Lagrange (1726-1813). The equations of motion are complex and highly nonlinear and can be only solved with the aid of computers. Most of the recent work [1-7] is therefore devoted to the methods that are efficient and general enough to be applicable to a wide variety of multibody systems with a minimum amount of preparatory work.

The two most popular methods for the derivation of equations of motion are the Lagrange's method and the Newton-Euler's method. As the final form of the equations derived by the Lagrange's method can also be obtained by means of Newton-Euler's method, it is only the latter method that will be considered.

In the Newton-Euler's method, equations of motion are written for each body of the system with internal reaction forces and torques appearing as external loads. Using graph theory concepts to describe the interconnections between the bodies and writing the equations of motion in a vector-matrix form it becomes evident that these equations can be viewed as a stationary point condition of a quadratic programming problem with equality constraints provided we replace the physical components of the hinge reaction forces/torques with the Lagrange multipliers. This point of view is useful since the theory of quadratic programming can be immediately applied for the formulation of the equations of motions and many seemingly different approaches can then be presented in a unified manner. In the literature [8,9], this quadratic programming problem is known as the Gauss's principle of least constraint.

The remainder of the report is divided into four sections. In Section 2, we present some concepts from graph theory and introduce the notion of incidence matrix to describe the interconnections between the bodies of a system with directed tree structure. Equations of motion are formulated for systems with tree structure in Section 3, first for the case when the hinges are ball-and-socket (spherical) joints and then for the case when the hinges are ball-and-socket joints, universal joints and pin joints. Several ways of formulating the equations of motion are discussed. They all stem from the same set of equations but differ in the way the Lagrangian matrix in the quadratic programming problem is manipulated. It is shown that equations formulated by the Lagrange's form of d'Alembert's principle [10] can also be formulated by using the Gauss's principle of least constraint. In Section 4, the methods of Section 3 are extended to multibody systems with closed chains. Conclusions are discussed in Section 5 and the matrix computational procedures necessary for the determination of linear accelerations, angular accelerations, and constraint forces and torques are presented in the Appendix.

2. DESCRIPTION OF THE INTERCONNECTIONS

Let us consider a system of n rigid bodies connected to each other by n-1 hinges. If we identify the bodies with the vertices and the hinges with the links, such a structure, in the language of graph theory, is called a nondirected tree [11]. Figs. 1a and 1b respectively show a 6-body system and its tree graph.

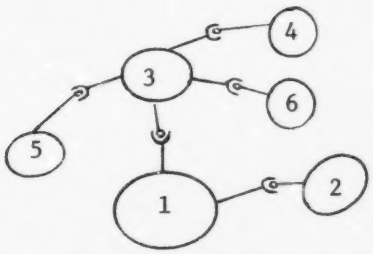

Fig. 1a A 6-body system with tree structure

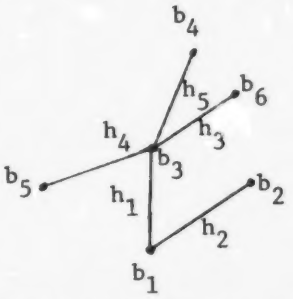

Fig. 1b System grpah of a 6-body system

In studying the dynamics of multibody systems the hinge forces and torques act on two contiguous bodies with opposite signs and hence we must specify unambiguously which of the two bodies is acted upon by force/torque with a positive sign and which one with a negative sign. This means that to describe the interconnections between the bodies we must convert a nondirected tree to a directed graph by assigning a sense of direction. To be specific we shall convert the nondirected tree into a directed tree. By this we mean a directed graph without a circuit for which the indegree of every vertex b_i, (i.e. the number of arcs which have b_i as their final vertex) except one is unity: the indegree of the exceptional vertex, called the root of the tree, being zero.

A nondirected tree can be converted into a directed tree by arbitrarily picking any vertex as the root and choosing directions for the links so that there is a path between the root and every other vertex. It should be noted that there can be only one such path. Fig. 2 shows a directed tree representation of the system shown in Fig. 1b.

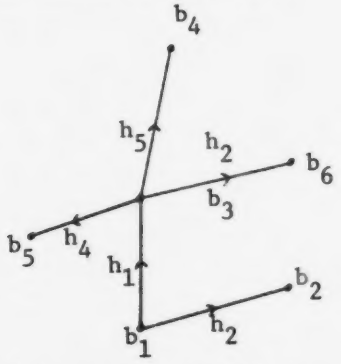

Fig. 2 Directed tree representation of the 6-body system

So far we have not paid any particular attention to the labeling of either the vertices or the arcs. The numbers were assigned in an arbitrary way. We now describe a procedure called regular labeling that produces a simple structure for the incidence matrix (defined below). For this, label the root as body 1 and the peripheral vertices the highest-numbers n, n-1,.... The peripheral vertices are those vertices which have no arcs emanating from them. The arc with final vertex n is labeled n-1 and that with n-1 is labeled n-2 and so on. Now all vertices and arcs that have been labeled are removed from the tree producing new peripheral vertices. The procedure is then repeated until all vertices and arcs have been labeled. The regular labeling and an arbitrary labeling corresponding to the system of Fig. 1a are shown in Figs. 3a and 3b respectively.

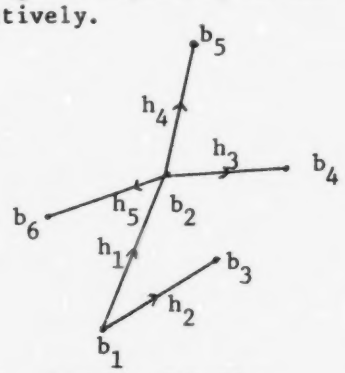

Fig. 3a Regular labeling

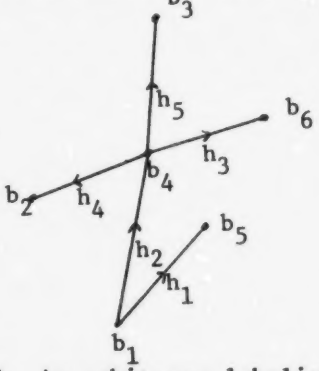

Fig. 3b An arbitrary labeling

A directed tree is conveniently represented by an nxn-1 incidence matrix C (of rank n-1) whose elements c_{ij} are defined as follows:

$c_{ij} = 1$ if b_i is the initial vertex of arc h_j

$c_{ij} = -1$ if b_i is the final vertex of arc h_j

$c_{ij} = 0$ otherwise

Since each arc is adjacent to exactly two vertices, each column of the incidence matrix contains one element 1 and one element -1. For the directed trees shown in Figs. 3a and 3b the incidence matrices C_a and C_b are

$$C_a = \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{array} \begin{array}{c} \begin{array}{ccccc} h_1 & h_2 & h_3 & h_4 & h_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \end{array} \qquad (1)$$

$$C_b = \begin{array}{c} \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{array} \begin{array}{c} \begin{array}{ccccc} h_1 & h_2 & h_3 & h_4 & h_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix} \end{array}$$

It should be noted that with the first row omitted the incidence matrix C_a corresponding to the regular labeling is upper triangular having all elements on the main diagonal as -1, while C_b does not have any such simple structure.

3. EQUATIONS OF MOTION FOR SYSTEMS WITH TREE STRUCTURE

Consider a system of n bodies with tree structure. Let us isolate the ith body from the system. The forces acting on the body are the external forces, hinge forces and the inertia forces. As regards the hinge forces and torques we adopt the following sign convention. The forces and torques at hinge j are taken positive on that vertex which forms the initial vertex of arc h_j. The index of such a body will be denoted by j^+. The index of the body which is the final vertex of arc h_j will be denoted by j^-. For example, for the system of Fig. 3a, $5^+ = 2, 5^- = 6$ and on body 2 the force is f_5 and torque g_5 while on body 6 the force is $-f_5$ and torque $-g_5$.

Let ℓ_{ij} denote the position vector from centre of mass C_i of body b_i to the hinge j, Fig. 4. Position vector ℓ_{ij} is, of course, zero if hinge

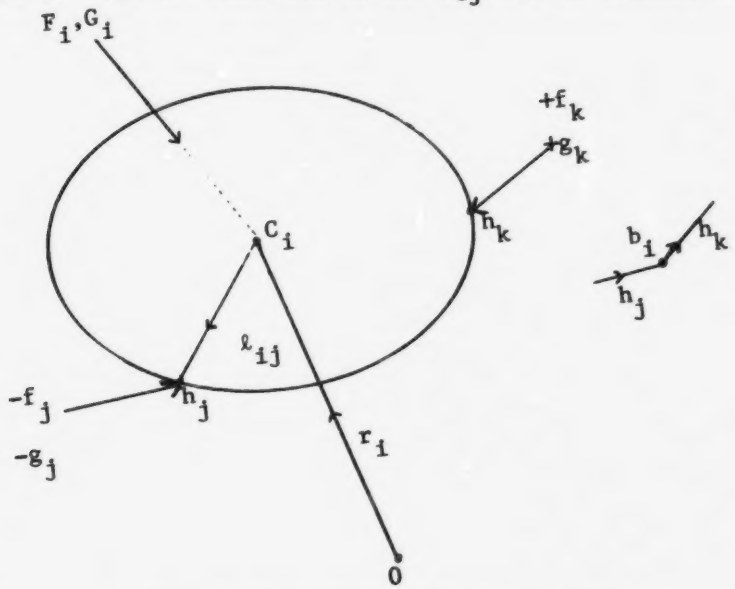

Fig. 4 Free-body diagram of body i

j is not located on body i. The equations of motion of of body i, i=1,2,...n, are:

$$m_i \ddot{r}_i = F_i + \sum_{j=1}^{n-1} c_{ij} f_j \quad \text{(Newton's Equations)} \tag{2}$$

$$I_i \dot{\omega}_i + \omega_i \times I_i \omega_i = R_i^T [G_i + \sum_{j=1}^{n-1} c_{ij} (R_i \ell_{ij} \times f_j + g_j)] \quad \text{(Euler's equations)} \tag{3}$$

where T denotes the transpose and

m_i = the mass of body i

I_i = the inertia matrix of body i about its centre of mass C_i

r_i = the position vector of C_i from an inertially fixed point 0

ℓ_{ij} = vector from C_i to hinge j, expressed in body frame i

F_i, G_i = the resultant external force and torque acting at C_i, expressed in an inertial frame

f_j, g_j = hinge force and torque at joint j, expressed in an inertial frame

ω_i = the absolute angular velocity of body i, expressed in body frame i

R_i = the transformation matrix connecting the ith frame to the inertial frame.

In the vector-matrix notation the equations of motion can be expressed as:

$$M \ddot{r} = F + Cf \tag{4}$$

$$\dot{h} = G + L \times f + Cg \tag{5}$$

where the various vectors and matrices are defined as follows:

M = diagonal matrix with diagonal elements m_i or $m_i E_3$ (E_3=3x3 unit matrix)

$\ddot{r} = (\ddot{r}_1, \ddot{r}_2, \ldots, \ddot{r}_n)^T$; F,G defined in the same way

$f = (f_1, f_2, \ldots, f_{n-1})^T$; g defined in the same way

C = nxn-1 matrix with elements c_{ij} or 3nx3(n-1) matrix with elements $c_{ij}E_3$

L = nxn-1 matrix with elements $L_{ij} = c_{ij}R_i\ell_{ij}$

$h = (h_1, h_2, \ldots, h_n)^T$ an n-vector with the ith element

$$h_i = R_i (I_i \dot{\omega}_i + \omega_i \times I_i \omega_i)$$

It should be noted that the elements of the matrix L and the vectors r, f, g, F, G and h are 3x1 matrices, while the elements of M and C can be taken either as scalars or 3x3 matrices.

Eqs. (4) and (5) can be used to describe the motion of the system provided the forces and torques at the couplings between two neighbouring bodies are known. Such is not the case when the hinges are ball-and-socket (spherical) joints giving arise to constraint forces that must be determined or eliminated. In case the hinges are universal joints or pin joints then in addition to the constraint forces there are also constraint torques.

3.1 Systems With Ball-and-Socket Joints

Let us first consider the simplest case when all hinges are ball-and-socket joints. The relative motion of any two neighbouring bodies is a pure rotation with three degrees of freedom. From Fig. 5, the constraint equation at hinge j is given by:

$$r_{j+} + R_{j+}\ell_{j+,j} = r_{j-} + R_{j-}\ell_{j-,j} \qquad j=1,2,\ldots,n-1$$

or

$$c_{j+,j}r_{j+} + c_{j-,j}r_{j-} + c_{j+,j}R_{j+}\ell_{j+,j} + c_{j-,j}R_{j-}\ell_{j-,j} = 0$$

or

$$\sum_{i=1}^{n} c_{ij} r_i + \sum_{i=1}^{n} c_{ij} R_i \ell_{ij} = 0 \qquad (6)$$

since for each j, $c_{j+,j}=1$, $c_{j-,j}=-1$ and $c_{ij}=0$, $i \neq j+$ or $j-$.

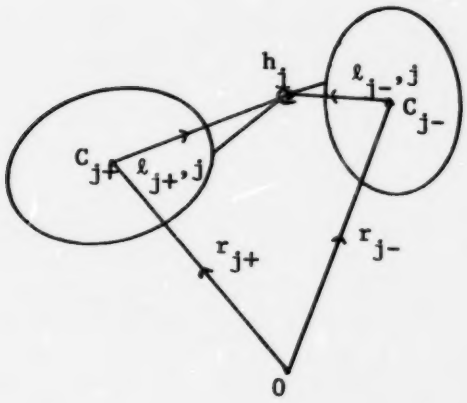

Fig. 5 Vectors depicting the constraint at hinge j

In vector-matrix form Eq. (6) can be written as

$$C^T r + L^T 1_n = 0 \tag{7}$$

where 1_n is an n-vector defined by $1_n = (1,1,\ldots,1)^T$. Differentiating Eq. (7) twice with regard to time we obtain

$$C^T \ddot{r} + \ddot{L}^T 1_n = 0 \tag{8}$$

where each element $\ddot{L}_{ij}$ of $\ddot{L}$ is given by

$$\ddot{L}_{ij} = c_{ij} R_i (\dot{\omega}_i \times \ell_{ij} + \omega_i \times (\omega_i \times \ell_{ij})) \tag{9}$$

Eqs. (4), (5) and (8) can be written in the matrix form as follows:

$$\begin{pmatrix} M & 0 & -C \\ 0 & I & -\tilde{L} \\ -C^T & -(\tilde{L})^T & 0 \end{pmatrix} \begin{pmatrix} \ddot{r} \\ \dot{\omega} \\ f \end{pmatrix} = \begin{pmatrix} F \\ G + Cg - \hat{h} \\ (\hat{L})^T 1_n \end{pmatrix} \triangleq \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \tag{10}$$

where each element in matrices M and C is a 3x3 diagonal matrix having that element along the diagonal; 0 is a matrix of appropriate dimensions with zero elements; and

$\hat{L}_{ij} \triangleq$ the ijth element of $\hat{L} = R_i(\omega_i \times (\omega_i \times c_{ij} \ell_{ij}))$

$\hat{h}_i$ = the ith component of $\hat{h} = R_i(\omega_i \times I_i \omega_i)$

$I = \text{diag}\{R_1 I_1 R_1^T, \ldots, R_n I_n R_n^T\}$

$\dot{\omega} = (R_1\dot{\omega}_1, \ldots, R_n\dot{\omega}_n)$

$\tilde{L}$ = n×n-1 matrix with elements $\tilde{L}_{ij}$, i=1,...,n, j=1,...,n-1, and the tilda (~) on a vector r with components x,y, and z denotes a 3×3 antisysmmetric matrix

$$\tilde{r} = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \tag{11}$$

so that $r_1 \times r_2 = \tilde{r}_1 r_2$. Eq. (10) represents 6n + 3(n-1) scalar equations for the determination of the linear accelerations, angular accelerations and the constraint forces. It should be noted that the matrix representing Eq. (10) is sparse (also symmetric) and hence sparse matrix techniques can be used to invert this matrix. We can also reduce the order of the matrix inversion problem through some preliminary analytic manipulations.

From Eq. (10) we see that if f is known then $\ddot{r}$ and $\dot{\omega}$ can be determined easily by inverting the diagonal matrices M and I:

$$\ddot{r} = M^{-1}(u_1 + Cf) \tag{12}$$

$$\dot{\omega} = I^{-1}(u_2 + \tilde{L}f) \tag{13}$$

Now, to determine f multiply the first row by $C^T M^{-1}$, the second row by $(\tilde{L})^T I^{-1}$ and add to the third. This yields

$$-(C^T M^{-1} C + (\tilde{L})^T I^{-1} \tilde{L})f = C^T M^{-1} u_1 + (\tilde{L})^T I^{-1} u_2 + u_3 \tag{14}$$

The 3(n-1)×3(n-1) matrix representing Eq. (14) can also be written as

$$-(C^T (\tilde{L})^T) \begin{pmatrix} M^{-1} & 0 \\ 0 & I^{-1} \end{pmatrix} \begin{pmatrix} C \\ \tilde{L} \end{pmatrix}$$

and can be inverted numerically (see Appendix) and f determined. Having found f, the linear and angular positions and velocities can be computed from Eqs. (12) and (13) by integration. We mention that only the first equation of Eq. (12) needs to be integrated since if the position vector r_1 is known the others can be determined from Eq. (7). Also knowing $\dot{r}_1$, the remaining $\dot{r}_i$ can be determined from the equation $C^T\dot{r} + \tilde{L}^T 1_n = 0$.

It should be noted that the above mentioned procedure requires an inversion of a $3(n-1)\times 3(n-1)$ matrix (Eq. (14)) and the inversions of n, 3×3 matrices (Eq. (13)).

We now discuss another matrix reduction procedure. We first note that the second equation in Eq. (10) does not contain $\ddot{r}$ term. So if we eliminate $\ddot{r}$ term from the third equation we obtain equations in f and $\dot{\omega}$. For this multiply the first equation of Eq. (10) by $C^T M^{-1}$ and add it to the third. This yields

$$\begin{pmatrix} I & -\tilde{L} \\ -(\tilde{L})^T & -C^T M^{-1} C \end{pmatrix} \begin{pmatrix} \dot{\omega} \\ f \end{pmatrix} = \begin{pmatrix} u_2 \\ C^T M^{-1} u_1 + u_3 \end{pmatrix} \tag{15}$$

At this stage we can either solve Eq. (15) numerically for $\dot{\omega}$ and f or reduce the dimensionality of the matrix further. Eliminating f we obtain

$$(I + \tilde{L}\,(C^T M^{-1} C)^{-1} (\tilde{L})^T)\dot{\omega} = -\tilde{L}(C^T M^{-1} C)^{-1}(C^T M^{-1} u_1 + u_3) + u_2 \tag{16}$$

Eq. (16) requires the inversion of the matrix $C^T M^{-1} C$, which can be obtained either numrically or analytically. Numerical inversion in the context of systems with tree structure having only one branch is discussed in [12]. We note that the matrix in this case is tridiagonal.

To obtain the matrix inversion analytically consider the matrix

$$\begin{pmatrix} M & -C \\ -C^T & 0 \end{pmatrix} \tag{17}$$

This matrix is a submatrix of the matrix representing Eq. (10) and can be thought of as arising from the optimization problem:

$$\min_{\ddot{r}} \frac{1}{2} (\ddot{r} - M^{-1}F)^T M (\ddot{r} - M^{-1}F)$$

subject to the constraints

$$C^T\ddot{r} + \ddot{L}^T 1_n = 0$$

This is nothing else but a quadratic programming problem with equality constraints and the inverse of the Lagrangian matrix (17) can be written in the following two alternative forms (see for example, [13]):

$$\begin{pmatrix} M & -C \\ -C^T & 0 \end{pmatrix}^{-1} = \begin{pmatrix} M^{-1} - M^{-1}CB & -M^{-1}CA \\ -B & -A \end{pmatrix} \tag{18}$$

$$\begin{pmatrix} M & -C \\ -C^T & 0 \end{pmatrix}^{-1} = \begin{pmatrix} E & EMT_2^T - T_2^T \\ T_2ME - T_2 & T_2MEMT_2^T - T_2M\,T_2^T \end{pmatrix} \tag{19}$$

where

$$B = A\,C^T M^{-1}$$

$$A = (C^T M^{-1} C)^{-1}$$

$$E = T_1^T (T_1 M T_1^T)^{-1} T_1$$

and T_1 is a $1 \times n$ matrix such that $T_1C=0$ and T_2 is an $n-1 \times n$ matrix such that $T_2C=E_{n-1}$, the unit $n-1 \times n-1$ matrix. In addition, matrix $T=\begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$ is nonsingular. Comparing Eqs. (18) and (19) we obtain

$$B = T_2 - T_2ME \tag{20}$$

$$A = -(T_2MEMT_2^T - T_2MT_2^T) \tag{21}$$

Thus A and B can be determined analytically provided the matrices T_1 and T_2 are known.

A general method for computing T_1 and T_2 is to augment the matrix C by adding a column such that the resulting matrix is nonsingular (see Appendix). In the present case we can take the following matrix as the augmented matrix

$$S = (-e_1\ C) \tag{22}$$

where $e_1 = (1\ 0\ 0\ \ldots\ 0)^T$. The matrix S is nonsingular since it is upper triangular with -1 elements along the main diagonal. Let its inverse be denoted by T and represented as a partitioned matrix

$$T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} \tag{23}$$

Then from the definition of the inverse we have $T_1C=0$, $T_2C=E_{n-1}$, $T_1e_1=-1$ and $T_2e_1=0$. Since each column of C contains one element 1 and one element -1 it follows from $T_1C=0$ and $T_1e_1=-1$, that $T_1=-1_n$ where 1_n is a column matrix of n elements, each equal to 1.

To determine T_2, first consider $T_2e_1=0$. Performing the matrix multiplication we see that the first column of T_2 is zero. Now using $T_2C=E_{n-1}$ we obtain for each j (j=1,2,...,n-1)

$$t_{ij-}=t_{ij+}-\delta_{ij}, \quad i=1,\ldots,n-1 \tag{24}$$

where δ_{ij} is the Kronecker symbol and t_{ij}'s are the elements of T_2.

Matrix T has a simple graph-theoretical interpretation [2]. Imagine a fictitious body b_o attached to the root b_1 of the directed tree such that b_o is the initial vertex of the arc h_o connecting b_o to b_1. The elements $t_{k\ell}$,(k=0,1,...,n-1; ℓ=1,2,...,n), of T are obtained from the relations

$t_{k\ell}$ = -1 if arc h_k is on the path between b_o and b_ℓ

= 0 otherwise

For the case of a multibody system represented by the graph in Fig. 3a we have

$$T=\left(\begin{array}{c|cccc} -1 & -1 & -1 & -1 & -1 & -1 \\ \hline 0 & -1 & 0 & -1 & -1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right) \tag{25}$$

Notice that matrix T is an upper triangular (since S is) with -1 as diagonal elements.

Substituting the value of T_1 into the expression for E, B and A we obtain

$$E=\frac{1}{m}\,1_n1_n^T,\quad B=T_2\mu,\quad A=T_2M\mu^TT_2^T \tag{26}$$

where m is the total mass of the system and $\mu=E_n-\frac{1}{m}M1_n1_n^T$. Since $\mu M\mu^T=M\mu^T$ the matrix A can also be written as

$$A=T_2\mu M\mu^TT_2^T=T_2\mu M(T_2\mu)^T$$

$$=BMB^T \tag{27}$$

Using Eq. (27), Eq. (16) can be written as

$$(I + \tilde{D}\, M(\tilde{D})^T)\, \dot{\omega} = \tilde{D}\, u_1 + u_2 + \tilde{D}\, M\, B^T u_3 \tag{28}$$

where the nxn matrix $D = -LB = -LT_2\mu$. Substituting the values of u_1, u_2, u_3 into Eq. (28), the rotational equations can be written as

$$(I + \tilde{DM}(\tilde{D})^T)\, \dot{\omega} = \tilde{DF} - \tilde{DM}(\hat{D})^T\, 1_n - \hat{h} + G + Cg \tag{29}$$

where $\hat{D}$ is defined in the same way as $\hat{L}$.

Matrix D introduced above has a simple physical interpretation. Augment the mass of each body i by placing point masses at each hinge on the body, the mass at hinge j being the sum of masses of all bodies connected to body i at hinge j either directly or indirectly. As an example, for the system shown in Fig. 3a, the augmented body 2 is obtained by placing the masses m_1+m_3, m_4, m_5 and m_6 at joints 1,3,4 and 5 respectively. It is clear from the definition that all augmented bodies have the same mass m, the total mass of the system. The centre of mass of the augmented body is called the barycenter [1,2] of the body. The diagonal elements d_{ii} is the vector from the barycenter of body i to the original centre of mass C_i while for each body k connected to body i, directly or indirectly at joint j, d_{ik} is the vector from the barycenter to joint j. It is quite clear that d_{ik} are not all different. For example (Fig. 3a), $d_{12} = d_{14} = d_{15} = d_{16}$.

To obtain some insight into the various terms involved in Eq. (29) let us evaluate the matrix $DM(D)^T$. The ith diagonal element is given by

$$-\sum_{k=1}^{n} m_k \tilde{d}_{ik} \tilde{d}_{ik} \tag{30}$$

which simply represents the inertia matrix of the point masses of the augmented body i about its barycenter. This means that the diagonal elements of $I+\tilde{DM}(\tilde{D})^T$ represent the inertia matrices of the augmented bodies. Similarly the ijth term of the matrix $DM(D)^T = -\sum_{k=1}^{n} m_k d_{ik} d_{jk}$. Using the properties of d_{ik} it can be shown [2] that

$$-\sum_{k=1}^{n} m_k \tilde{d}_{ik} \tilde{d}_{jk} = m_j \tilde{d}_{ij} \tilde{d}_{ji} \tag{31}$$

where j here refers to the index of the body.

We now simplify the expressions appearing in the matrix $-\tilde{DM}(\hat{D})^T$. The ith diagonal element of this matrix is

$$-\sum_{k=1}^{n} m_k \tilde{d}_{ik} \hat{d}_{ik} = -\sum_{k=1}^{n} m_k \tilde{d}_{ik} (R_i \omega_i \times (R_i \omega_i \times d_{ik})) \tag{32}$$

Using the identity $\tilde{a} \times (\tilde{b} \times (\tilde{b} \times \tilde{c})) = -\tilde{b}\tilde{c}\tilde{a}\tilde{b} + \tilde{c}\tilde{a}\tilde{b}^T\tilde{b}$, Eq. (32) can be written as

$$-\sum_{k=1}^{n} m_k \tilde{d}_{ik} \hat{d}_{ik} = (R_i \tilde{\omega}_i)(\sum_{k=1}^{n} m_k \tilde{d}_{ik} \tilde{d}_{ik}) R_i \omega_i \tag{33}$$

Similarly the ijth element of the matrix is

$$\begin{aligned} -\sum_{k=1}^{n} m_k \tilde{d}_{ik} \hat{d}_{jk} &= (R_j \tilde{\omega}_j)(\sum_{k=1}^{n} m_k \tilde{d}_{jk} \tilde{d}_{ik}) R_j \omega_j - (\sum_{k=1}^{n} m_k \tilde{d}_{jk} d_{ik}) \tilde{\omega}_j^T \omega_j \\ &= -m(R_j \omega_j)\tilde{} \; \tilde{d}_{ji} \tilde{d}_{ij} R_j \omega_j + m \tilde{d}_{ji} d_{ij} \tilde{\omega}_j^T \omega_j \end{aligned} \tag{34}$$

Let K denote the positive definite matrix $I + \tilde{D}M(\tilde{D})^T$. Then from Eqs. (30) and (31) we have

$$\begin{aligned} K_{ii} &= R_i I_i R_i^T - \sum_{k=1}^{n} m_k \tilde{d}_{ik} \tilde{d}_{ik} \\ K_{ij} &= m_j \tilde{d}_{ij} \tilde{d}_{ji} \qquad i \neq j \end{aligned} \tag{35}$$

With this notation, Eqs. (33) and (34) can be written as

$$-\sum_{k=1}^{n} m_k \tilde{d}_{ik} \hat{d}_{ik} = (R_i \tilde{\omega}_i)(R_i I_i R_i^T - K_{ii}) R_i \omega_i \tag{36}$$

$$-\sum_{k=1}^{n} m_k \tilde{d}_{ik} \hat{d}_{jk} = -(R_j \tilde{\omega}_j) K_{ji} R_j \omega_j + m_j \tilde{d}_{ji} d_{ij} \tilde{\omega}_j^T \omega_j \tag{37}$$

Substituting these values into Eq. (29) we get

$$K\dot{\omega} = -\hat{h}' + \tilde{D}F + G + Cg + N 1_n \stackrel{\Delta}{=} \tilde{u}_2 \tag{38}$$

where

$\hat{h}'_i$ = the ith element of $\hat{h}' = (R_i \tilde{\omega}_i) K_{ii} (R_i \omega_i)$

and the matrix N is defined as follows

$N_{ij} = 0 \quad i=j$

$N_{ij} = -(R_j \omega_j)\tilde{} \; K_{ji} R_j \omega_j + m_j \tilde{d}_{ji} d_{ij} \tilde{\omega}_j^T \omega_j \quad i \neq j$

To determine $\dot{\omega}$ from Eq. (29) or Eq. (38) we can use the Cholesky's decomposition method for solving systems of linear equations with positive definite matrices. It should be noted that matrix K is not sparse.

Remark 1 It was mentioned above that the first and third equation of Eq. (10) can be obtained from an optimization problem. In fact, by using Lagrange multipliers method it can be seen that Eq. (10) is the stationary point condition of the following optimization problem:

$$\min_{\ddot{r},\dot{\omega}} \left\{ \frac{1}{2} (\ddot{r} - M^{-1}F)^T M(\ddot{r}-M^{-1}F) + \frac{1}{2} (\dot{\omega} - I^{-1}(G+Cg - \hat{h}))^T I(\dot{\omega}-I^{-1}(G+Cg-\hat{h}) \right\} \tag{39}$$

subject to the constraints $C^T\ddot{r} + \ddot{L}^T 1_n = 0$

or $$C^T\ddot{r} + (\tilde{L})^T\dot{\omega} = -(\hat{L})^T 1_n$$

This minimization problem is the statement of a principle known as Gauss's principle of least constraint [8,9] and can be exploited for the approximate determination of constraint forces (vector Lagrange multipliers) by using penalty function methods [14].

Remark 2 Equations of motion can also be obtained without the introduction of Lagrange multipliers. Let $\ddot{q}$ denote the (n+1) vector representing the independent variables for the optimization problem. Differentiating the optimizing function partially with regard to q, the stationary point condition is

$$\left(\frac{\partial \ddot{r}}{\partial \ddot{q}}\right)^T (M\ddot{r} - u_1) + \left(\frac{\partial \dot{\omega}}{\partial \ddot{q}}\right)^T (I\dot{\omega} - u_1) = 0$$

As $\dot{r}$ and ω can be taken as a linear combination of $\dot{q}$ it follows that

$\frac{\partial \ddot{r}}{\partial \ddot{q}} = \frac{\partial \dot{r}}{\partial \dot{q}}$ and $\frac{\partial \dot{\omega}}{\partial \ddot{q}} = \frac{\partial \omega}{\partial \dot{q}}$. Substituting these into the above equation we obtain

$$\left(\frac{\partial \dot{r}}{\partial \dot{q}}\right)^T (M\ddot{r} - u_1) + \left(\frac{\partial \omega}{\partial \dot{q}}\right)^T (I\dot{\omega}-u_2) = 0 \tag{40-a}$$

Eq. (40-a) is a vector-matrix formulation of the so-called Lagrange's form of d'Alembert's principle [10]. Eq. (40-a) can also be written as

$$\left[\left(\frac{\partial \dot{r}}{\partial \dot{q}}\right)^T, \left(\frac{\partial \omega}{\partial \dot{q}}\right)^T\right] \left[\begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \ddot{r} \\ \dot{\omega} \end{pmatrix} - \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right] = 0 \tag{40-b}$$

We now establish the connection between the equations of motion derived by using Eq. (40-b) and the Lagrange multipliers method. For this let $Z^T = (Z_1^T, Z_2^T)$ denote a matrix such that

$$Z^T\begin{pmatrix} C \\ \tilde{} \\ L \end{pmatrix} = 0$$

Since $\begin{pmatrix} C \\ \tilde{} \\ L \end{pmatrix}$ is of full rank $3(n-1)$, a nonzero Z^T of rank $3(n+1)$ exists.

Multiplying the first two equations of Eq. (10) by Z^T we obtain

$$(Z_1^T, Z_2^T)[\begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \dot{r} \\ \dot{\omega} \end{pmatrix} - \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}] = 0 \qquad (40\text{-c})$$

As Eqs. (40-b) and (40-c) are both independent of the Lagrange multipliers we can identify $\frac{\partial \dot{r}}{\partial \dot{q}}$ with Z_1 and $\frac{\partial \omega}{\partial \dot{q}}$ with Z_2. Thus to apply the Lagrange's form of d'Alembert's principle all we need is to determine a matrix Z^T. As an example, we can take

$$Z^T = \begin{pmatrix} 1_n^T & 0 \\ \tilde{} & \\ -LT_2 & E_n \end{pmatrix} \qquad (40\text{-d})$$

which corresponds to the choice of taking $\dot{q}_1 = \dot{r}_1, \dot{q}_{i+1} = \omega_i (i=1,\ldots,n)$. This can be easily seen since $\dot{r} = \dot{r}_1 1_n - (\tilde{LT_2})^T \omega$. Eliminating r by using Eq. (19) it can be seen that the rotational equations are the same as those given by Eq. (28). It should be noted that in the language of linear algebra the space generated by the columns of Z forms the orthogonal complement of the space generated by the columns of $\begin{pmatrix} C \\ \tilde{} \\ L \end{pmatrix}$.

Remark 3. In certain applications a multibody system is connected to an external body whose motion is prescribed. For example, a manipulator attached to a moving platform or to the ground. In such cases the above analysis can be easily modified.

We assume, without loss of generality, that the system is connected to the external body through a single hinge. For if there are more than one hinges, then the system can be broken into several dynamically independent systems. Label the external body as 0 and the body to which it is attached as 1. Let h_o denote the hinge connecting these two bodies. Then the additional constraint at hinge 0 can be written as

$$r_o - r_1 + (R_o \ell_{oo} - R_1 \ell_{10}) = 0 \qquad (41)$$

where r_o is the vector from the origin of the inertial frame to the centre of mass of body o and ℓ_{io} (i=0, 1) are the vectors from C_i to hinge 0. By defining $c_{io} = -1$ for i=1 and zero otherwise, Eq. (41) can also be rewritten as

$$-e_1^T r + \sum_{i=1}^{n} c_{io} R_i \ell_{io} = -\hat{r}_o$$

where $\hat{r}_o = r_o + R_o \ell_{oo}$. Combining this equation with the constraint Eq. (7) we obtain

$$S^T r + \bar{L}^T l_n = -\hat{r}_o e_1$$

where

$\bar{L}$ = nxn matrix with elements $\bar{L}_{ij} = c_{ij} R_i \ell_{ij}$; i=1,,,,n, j=0,1,...,n-1

Equations of motion can therefore be obtained by replacing C with S, L with $\bar{L}$ and by adding $\dot{\hat{r}}_o e_1$ to the right hand side of the third equation in Eq. (10). S is nonsingular, matrices B and A in this case are given by $B = T$ and $A = TMT^T$.

3.2 Systems with Ball-and-Socket Joints, Universal Joints and Pin Joints

We now discuss the case when the rotational degrees of freedom at some joints may be less than three i.e. the hinges may be universal (two degrees of freedom) or pin (one degree of freedom) joints. Let Ω_j denote the relative angular velocity of body j^- with respect to body j^+ expressed in the inertial frame. Then

$$\Omega_j = R_{j-} \omega_{j-} - R_{j+} \omega_{j+} = \sum_{i=1}^{n} c_{ij} R_i \omega_i, \quad j=1,...,n-1 \tag{42}$$

In terms of the relative angular rates, Ω_j can be written as

$$\Omega_j = \sum_{k=1}^{n_j} p_{jk} \dot{\gamma}_{jk} = P_j \dot{\gamma}_j \tag{43}$$

where $n_j = 1,2$, or 3 according as the hinge is a pin, universal or ball-and-socket joint and p_{jk} (k=1,...,n_j) denote the unit vectors along the axes of rotation and are functions of the orientation angles.

To obtain the rotational cosntraint conditions we denote by q_{jk} (k=1,...,n'_j) the unit vectors in the constraint directions where n'_j satisfies $n_j + n'_j = 3$. The vectors p_{jk} and q_{jk} are mutually orthogonal. Since there is no relative angular velocity in the direction of the constraint axes we have

$$Q_j^T \Omega_j = 0 \quad j=1,2,\ldots,n-1 \tag{44}$$

where Q_j is a matrix with elements q_{jk}. If Ω denotes the vector $(\Omega_1 \ldots, \Omega_{n-1})$ and Q the quasi-diagonal matrix $Q = \text{diag}\ \{Q_1,\ldots,Q_{n-1}\}$ then Eqs. (42) and (44) can be rewritten as

$$\Omega = -C^T\omega \tag{45}$$

$$Q^T \Omega = 0 \tag{46}$$

Substituting Eq. (45) into Eq. (46) we obtain the constraint conditions

$$Q^T C^T \omega = 0 \tag{47}$$

Equations of motion can now be written down as the stationary point condition of the optimization problem stated in Eq. (39) along with the additional constraints

$$Q^T C^T \dot{\omega} + \dot{Q}^T C^T \omega = 0 \tag{48}$$

where Eq. (48) is obtained by differentiating Eq. (47) with respect to time. Introducing the vector Lagrange multiplier $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_{n-1})$ where each λ_j is an n_j' vector we have

$$\begin{pmatrix} M & 0 & -C & 0 \\ 0 & I & -\tilde{L} & -CQ \\ -C^T & -(\tilde{L})^T & 0 & 0 \\ 0 & -Q^TC^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \ddot{r} \\ \dot{\omega} \\ f \\ \lambda \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \tag{49}$$

where u_1, u_2, u_3 are the same as in Eq. (10) and $u_4 = \dot{Q}^T C^T \omega$. Matrix representing Eq. (49) is sparse and symmetric and hence sparse matrix techniques can be used for solving Eq. (49). As in the case of equations without rotational constraints some preliminary matrix manipulations may be useful. For example, corresponding to Eq. (14) we have

$$-\begin{pmatrix} (C^TM^{-1}C + (\tilde{L})^TI^{-1}\tilde{L}) & (\tilde{L})^TI^{-1}CQ \\ Q^TC^TI^{-1}\tilde{L} & Q^TC^TI^{-1}CQ \end{pmatrix} \begin{pmatrix} f \\ \lambda \end{pmatrix} =$$

$$\begin{pmatrix} C^TM^{-1}u_1 + (\tilde{L})^TI^{-1}\ u_2 + u_3 \\ Q^TC^TI^{-1}u_2 + u_4 \end{pmatrix}$$

Several other possibilities can be explored for the solution of Eq. (49). As an example let us eliminate $\ddot{r}$, f from Eq. (49). This yields (see Eq. (38)) the following equation:

$$\begin{pmatrix} K & -CQ \\ -Q^T C^T & 0 \end{pmatrix} \begin{pmatrix} \dot{\omega} \\ \lambda \end{pmatrix} = \begin{pmatrix} \bar{u}_2 \\ u_4 \end{pmatrix} \tag{51}$$

Eq. (51) can now be solved numerically using some factorization method. The square matrix representing Eq. (51) has dimension $3n + \sum_{j=1}^{n-1} n'_j$ while the matrix representing Eq. (49) has dimension $6n + 3(n-1) + \sum_{j=1}^{n-1} n'_j$. However, it should be noted that matrix in Eq. (51) is less sparse than that in Eq. (49).

Rather than solving Eq. (51) numerically we can eliminate λ analytically by utilizing the condition that the constraint torque along the axis of rotation is zero. From Eq. (51) we have

$$K\dot{\omega} = \bar{u}_2 + C\, g^c \tag{52}$$

where

$$g^c = Q\lambda \tag{53}$$

is the constraint torque at the hinges. Let the quasi-diagonal matrix P be defined as

$$P = \text{diag}\,\{P_1,\ldots,P_{n-1}\} \tag{54}$$

where P_j, the jth diagonal element, is given by $P_j = (p_{j1},\ldots,p_{jn_j})$. P is $3(n-1)\times \sum_{j=1}^{n-1} n_j$ matrix and satisfies, by definition, the condition $P^T Q = 0$. Multiplying Eq. (53) on the left by P^T and using $P^T Q=0$ we obtain

$$P^T g^c = 0 \tag{54}$$

Eq. (54) is just a mathematical representation of the fact that the constraint torques along the axes of rotation are zero.

To utilize Eq. (54) we need to know the expression for g^c. For this multiply Eq. (52) on the left by the matrix T. This yields

$$T_1 K\dot{\omega} = T_1 \bar{u}_2 \tag{55}$$

$$T_2 K\dot{\omega} = T_2 \bar{u}_2 + g^c \tag{56}$$

since $T_1C=0$, $T_2C=E_{n-1}$. To eliminate the constraint torque from Eq. (56) we multiply it on the left by P^T and obtain

$$P^T T_2 K\dot{\omega} = P^T T_2 \bar{u}_2 \tag{57}$$

All that remains to be done is to express Eqs. (55) and (57) in terms of the relative joint angles. Substituting Eq. (43) into Eq. (45) and using the definition of P we have

$$-C^T\omega = P\,\dot{\gamma} \tag{58}$$

Multiplication on the left by T_2^T yields

$$-(CT_2)^T\omega = T_2^T P\,\dot{\gamma} \tag{59}$$

Since $CT_2 = E_n - e_1 1_n^T$ we obtain after some simplifications

$$\omega = \omega_1 1_n - T_2^T P\,\dot{\gamma} \tag{60}$$

By substituting Eq. (60) into Eqs. (55) and (57) we obtain the rotational equations of motion:

$$\begin{aligned} &1_n^T K 1_n \dot{\omega}_1 - 1_n^T K T_2 P\ddot{\gamma} = 1_n^T \bar{u}_2 + 1_n^T K T_2 P\dot{\gamma} \\ &-P^T T_2 K 1_n \dot{\omega}_1 + P^T T_2 K T_2 P\ddot{\gamma} = -P^T T_2 (\bar{u}_2 + K T_2 P\dot{\gamma}) \end{aligned} \tag{61}$$

Eq. (61) represents $3 + \sum_{j=1}^{n-1} n_j$ scalar equations corresponding to as many rotational degrees of freedom.

Remark 4. When the linear or angular velocities are determined by integrating a system of equations greater than the number of degrees of freedom then it is advisable [9,15] to replace the constraint Eqs. (8) and (48) by equations which are asymptotically stable. Consider, as an example, the constraint Eq. (48). If ω is chosen according to

$$\nu = Q^T C^T \omega = 0$$

then $\dot{\nu}=0$ (Eq. (48)), implies, at least theoretically, that $\nu = 0$ for all time t. However, due to unavoidable numerical errors during the computation the equation $\dot{\nu} = 0$ will not yield $\nu=0$ for all t. To correct this situation we replace $\dot{\nu}=0$ by $\dot{\nu} + \alpha\nu = 0$ ($\alpha > 0$) which is asymptotically stable so that any error in ν tends to zero with the passage of time.

In a similar manner Eq. (8) should be replaced by

$$\ddot{\xi} + \alpha\dot{\xi} + \beta\xi = 0 \quad (\alpha,\beta \quad 0)$$

where $\xi = C^T r + L^T 1_n$

Remark 5. The derivation of equations of motion have been discussed in connection with hinges that allow only rotational degrees of freedom between neighbouring bodies. This is not a restriction and the procedures can be extended in an obvious manner to other types of joints which also have translational degrees of freedom. As an example, consider the case of prismatic joint allowing translational motion along a unit axis $\bar{p}$. We can now find two mutually perpendicular directions $\bar{q}_{j1}, \bar{q}_{j2}$ such that $\bar{q}_{j1}^T \bar{p}_j = 0$, $\bar{q}_{j2}^T \bar{p}_j = 0$. Writing $s_j \bar{p}_j = -\sum_{i=1}^{n} (c_{ij} r_i + L_{ij})$ where s_j is the distance along $\bar{p}_j$ from a point in body j+, we obtain

$$\bar{q}_{j1}^T \left(\sum_{i=1}^{n} (c_{ij} r_i + L_{ij})\right) = 0$$

$$\bar{q}_{j2}^T \left(\sum_{i=1}^{n} (c_{ij} r_i + L_{ij})\right) = 0$$

These two equations should therefore replace the three scalar equations $\sum_{i=1}^{n} (c_{ij} r_i + L_{ij}) = 0$ in forming the translational cosntraint equations. Thus instead of Eq. (7) we have

$$\bar{Q}^T (C^T r + L^T 1_n) = 0$$

where $\bar{Q} = \text{diag}\{\bar{Q}_1, \ldots, \bar{Q}_{n-1}\}$ and $\bar{Q}_j = (\bar{q}_{j1}, \bar{q}_{j2})$. Since there is no rotational degrees of freedom we have $\Omega_j = 0$ i.e. $n_j = 0$ and Q_j in Eq. (44) can be taken as E_3.

4. EQUATIONS OF MOTION FOR SYSTEMS WITH CLOSED CHAINS

We now consider multibody systems with closed chains, that is, systems of n rigid bodies with (n-1)+n* hinges, where n* is a positive integer. Fig. 6 illustrates a 6-body system with 7 hinges. A closed chain system can be

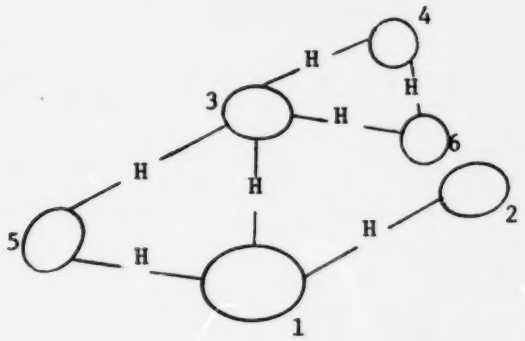

Fig. 6 A 6-body system with closed chains

converted into a system with tree structure in several ways by removing n* hinges. For example, for the system shown in Fig. 6 if we remove the hinges between bodies 1 and 5, and bodies 4 and 6 we obtain a system with tree structure.

To describe the interconnection structure for systems in closed chains we first determine the incidence matrix C for the system with tree structure by cutting n^* hinghes. The cut hinges are now labeled in an arbitrary order from n to $(n-1)+n^*$ and the arc direction chosen arbitrarily. In this way the $n\times(n-1)+n^*$ incidence matrix C_c for the closed chain can be written as

$$C_c = (C, C^*) \tag{62}$$

where the $n\times n-1$ matrix C represents the incidence matrix for the tree structure and the $n\times n^*$ matrix C^* represents the contribution from the cut hinges. For the system shown in Figure 6, the dimensions of the matrices C_c, C and C^* respectively are 6x7, 6x5 and 6x2. It should be noted that the rank of the incidence matrix C_c is n-1, the same as that of C.

Equations of motion can be written down in exactly the same manner as is done for the case of systems with tree structure. For example, in the case of ball-and-socket joints we have

$$\begin{pmatrix} M & 0 & -C_c \\ 0 & I & -L_c \\ -C_c^T & -(L_c)^T & 0 \end{pmatrix} \begin{pmatrix} \ddot{r} \\ \dot{\omega} \\ \lambda_c \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \tag{63}$$

where $\lambda_c=(\lambda_1,\ldots,\lambda_{n-1+n^*})^T$ is a vector Lagrange multiplier with each λ_i a 3-vector. The Lagranian matrix in Eq. (63) is nonsingular if and only if the matrix H,

$$H=\begin{pmatrix} C_c \\ \sim \\ L_c \end{pmatrix} \tag{64}$$

is of rank $3(n-1+n^*)$. If rank of H is $3(n-1+n^*)$ we use the procedures discussed in Section 3 to obtain the final form of the equations of motion. Otherwise we must use those procedures (see Appendix) which do not require H to be of rank $3(n-1+n^*)$. Note that the rank deficiency implies that the constraints are not independent.

We can also make use of the results of Section 3 in another way by partioning the constraint equation in two parts, one corresponding to the tree structure and the other due to the cut hinges and writing only the equations of motion for the tree structure. Let the constraint equation be written as

$$C^T r + L^T 1_n = 0 \tag{65}$$

$$C^{*T} r + L^{*T} 1_n = 0 \tag{66}$$

where the unstarred Eq. (65) corresponds to the constraints for the system reduced to tree structure. Using the constraint Eq. (65) we can write the equations of motion as

$$\begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \ddot{r} \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} C \\ \sim \\ L \end{pmatrix}\lambda + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{67}$$

Eliminating λ by multiplying on the left by the matrix Z^T (see Remark 2).

$$Z^T=\begin{pmatrix} 1_n^T & 0 \\ \sim & \\ -LT_2 & E_n \end{pmatrix}$$

we obtain a system of (n+1) vector equations:

$$\begin{pmatrix} 1_n^T M & 0 \\ \sim & \\ -LT_2 M & I \end{pmatrix}\begin{pmatrix} \ddot{r} \\ \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} 1_n^T u_1 \\ \sim \\ -LT_2 M u_1 + u_2 \end{pmatrix} \tag{68}$$

Eq. (68) can be easily expressed in terms of $\ddot{r}_1$ and $\dot{\omega}$. By multiplying Eq. (7) on the left by T_2^T we have $r = r_1 1_n - (LT_2)^T 1_n$. Differentiating this equation twice with regard to t and substituting it into the first equation of Eq. (68) we obtain

$$m\,\ddot{r}_1 - 1_n^T M(\tilde{LT}_2)^T\dot{\omega} = 1_n^T u_1 + 1_n^T M(\hat{LT}_2)^T 1_n \tag{69}$$

Using Eq. (19) we can express $\ddot{r}$ as

$$\ddot{r} = Eu_1 - (T_2\mu)^T[(\tilde{L})^T\dot{\omega} + (\hat{L})^T 1_n]$$

Substituting the above equation into the second equation of Eq. (68) we obtain Eq. (28). Combining Eq. (69) and Eq. (28) into one matrix equation we have

$$\begin{pmatrix} m & -1_n^T M(\tilde{LT}_2)^T \\ 0 & I + \tilde{DM}(\tilde{D})^T \end{pmatrix}\begin{pmatrix} \ddot{r}_1 \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} 1_n^T u_1 + 1_n^T M(\hat{LT}_2)^T 1_n \\ \tilde{D}u_1 + u_2 + \tilde{DMB}^T u_3 \end{pmatrix} \tag{70}$$

These are the equations of motion for the system with tree structure. To obtain the equations for the original system we notice that we have some further constraints in the form of Eq. (66). Differentiating this equation twice with respect to time and substituting the expression for $\ddot{r}$ in terms of $\ddot{r}_1$ and $\dot{\omega}$ we have

$$C^{*T}[\ddot{r}_1 1_n - (\tilde{LT}_2)^T\dot{\omega} - (\hat{LT}_2)^T 1_n] + (\tilde{L}^*)^T\dot{\omega} = -(\hat{L}^*)^T 1_n \tag{71}$$

where the starred quantities have the same meanings as the unstarred ones. By using the notation $\ddot{q} = (\ddot{r}_1, \dot{\omega}^T)^T$, $H^T = (C^{*T}1_n, (\tilde{L}^*)^T - C^{*T}(\tilde{LT})_2^T$ and $\bar{u}_3 = C^{*T}(LT_2)^T 1_n - (\hat{L}^*)^T 1_n$, Eq. (71) can be rewritten as

$$H^T\ddot{q} = \bar{u}_3 \tag{72}$$

Introducing the vector Lagrange multiplier λ^* the equations of motion can be obtained as a solution of the following matrix equation:

$$\begin{pmatrix} J & -H \\ -H^T & 0 \end{pmatrix}\begin{pmatrix} \ddot{q} \\ \lambda^* \end{pmatrix} = \begin{pmatrix} \bar{u} \\ -\bar{u}_3 \end{pmatrix} \tag{73}$$

where J and $\bar{u}$ respectively denote the matrix and the right hand side of Eq. (70). As the form of Eq. (73) is the same as before, any of the previously mentioned method can be used for its solution. The exception, of course, is when the rank of H is not $3n^*$ implying that the constraints given by Eq. (72) are not independant. We must therefore remove the redundant constraints from Eq. (72) first and then apply the methods of section 3 for formulating the equations of motion. Alternatively we can solve Eq. (73) by using those methods that do not require that matrix H be of full rank $3n^*$.

5. CONCLUSIONS

Several computer-oriented methods for the formulation of the equations of motion for multibody systems have been presented in a unified manner. Starting with the case of a multibody system with tree structure and with ball-and-socket joints, and using Newton-Euler's method it became evident that the equations of motion written in the vector-matrix form can be viewed as the stationary point condition of a quadratic programming problem with equality constraints. In fact, this QP problem is a statement of a not too well-known principle called the Gauss's principle of least constraint which is applicable to both holonomic and nonholonomic systems. Using QP theory it is shown that the various methods are nothing else but different ways of solving the Lagrangian system of equations in the Lagrangian method. Procedures are also discussed for systems with joints other than ball-and-socket joints and for systems with closed chains. However, due to the lack of computational results, no comparisons among the various methods are made.

6. REFERENCES

1. Hooker, W.W., "A Set of r Dynamical Attitude Equations for an Arbitrary n-Body Satellite Having r Rotational Degrees of Freedom", AIAAJ., Vol. 8, 1970, pp. 1205-1207.

2. Wittenburg, J., "Dynamics of Systems of Rigid Bodies", B.G. Teubner, Stuttgart, 1977.

3. Uicker, J.J., "Dynamic Behaviour of Spatial Linkages", Journal of Engineering for Industry, Trans. ASME, Series B, Vol. 91, 1969, pp. 251-265.

4. Chace, M.A., and Sheth, P.N., "Adaptation of Computer Techniques to the Design of Mechanical Dynamic Machinery," ASME Paper 73-DET-58, 1973.

5. Huston, R.L., and Passerello, C., "On Multi-Rigid-Body System Dynamics", Journal Computers and Structures, Vol. 10, 1979, pp. 439-446.

6. Vukobratovic, M., and Potkonjak, V., "Contribution of the Forming of Computer Methods for Automatic Modelling of Spatial Mechanisms Motions", Mechanism and Machine Theory, Vol. 14, 1979, pp. 179-188.

7. Paul, B., "Analytical Dynamics of Mechanisms - A Computer Oriented Overview," Mechanism and Machine Theory, Vol. 10, 1975, pp. 481-507.

8. Routh, E.J., "Dynamics of Systems of Rigid Bodies" (Elementary Part), Dover Publications, New York, 1960.

9. Roberson, R.E. "Constraint Stabilization for Rigid Bodies: An Extension of Baumgarte's Method", Magnus, K., Ed., Dynamics of Multibody Systems, Springer-Verlag, New York, 1978, pp. 274-289.

10. Kane, T.R., "Dynamics of Nonholonomic Systems", J. Applied Mech. Vol. 28, 1961, pp. 574-578.

11. Christofides, N., "Graph Theory: An Algorithmic Approach", Academic Press, New York, 1975.

12. Mufti, I.H., "Dynamics of Multi-Body Systems", NRC, DME MK-29, National Research Council Canada, Ottawa, November 1979.

13. Fletcher, R., "Practical Methods of Optimization, Volume 2, Constrained Optimization", John Wiley and Sons, New York, 1981.

14. Mufti, I.H., "On the Dynamic Equations of Manipulators", Proc. 1982 Canadian Conference on Industrial Computer Systems, 81-1-81-9, McMaster University, Hamilton, Ontario, May 1982.

15. Baumgarte, J., "Stabilization of Constraints and Integrals of Motion in Dynamical Systems," Computer Methods in Appl. Mech. and Eng. Vol. 1, 1972, pp. 1-16.

16. Jennings, A., "Matrix Computation for Engineers and Scientists", John Wiley and Sons, New York, 1977.

17. Forsythe, G.E., Malcolm, M.A., and Moler, C.B., "Computer Methods for Mathematical Computations", Prentice-Hall, Englewood Cliffs, 1977.

18. Beresford, P.J., "Sparse Matrix Techniques in Engineering Practice", in Sparse Matrices and Their Uses, (I.S. Duff ed.), Academic Press, New York, 1981, pp. 175-189.

19. George, A., and Liu, J.W., "Computer Solution of Large Sparse Positive Definite Systems", Prentice-Hall, Englewood Cliffs, 1981.

APPENDIX

In this Appendix we shall consider the solutions of systems of equations with matrices of the following two forms:

$$H^T J^{-1} H \tag{A-1}$$

$$Z^T J Z \tag{A-2}$$

where H and Z respectively are M x N and M x M-K matrices such that $Z^T H = 0$, K is the rank of matrix H, and J is an MxM positive definite matrix. The symmetric matrices (A-1) and (A-2) arise in the formulation of the equations of motion of multibody system. For example, in Eq. (14)

$$H = \begin{pmatrix} C \\ L \end{pmatrix}, \quad J = \begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix},$$

and in Eq. (40-c)

$$Z = \begin{pmatrix} 1_n & -(\tilde{LT}_2)^T \\ 0 & E_n \end{pmatrix}, \quad J = \begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix}$$

We need to consider two cases -(1) rank of H is N and (2) rank of $H < N$.

Case 1, K=N

In this case the matrix $H^T J^{-1} H$ is positive definite and the usual method for solving a system of equations with symmetric positive definite matrix is the Cholesky factorization with sparsity taken into consideration [16-19]. Let J be factored as $J = LDL^T$ where L is a unit lower triangular matrix and D is a diagonal matrix with positive diagonal elements. Substituting this value of J into $H^T J^{-1} H$ we obtain

$$\begin{aligned} H^T J^{-1} H &= H^T (LDL^T)^{-1} H \\ &= H^T L^{-T} D^{-\frac{1}{2}} D^{-\frac{1}{2}} L^{-1} H \end{aligned} \tag{A-3}$$

The Cholesky factorization is now obtained by performing the QR factorization of the matrix $D^{-\frac{1}{2}} L^{-1} H$:

$$D^{-\frac{1}{2}} L^{-1} H = Q \begin{pmatrix} R \\ 0 \end{pmatrix} \tag{A-4}$$

where Q is an MxM orthogonal matrix and R is an NxN upper triangular matrix. Substituting Eq. (A-4) into (A-3) we obtain

$$H^T J^{-1} H = (R^T \; 0)\, Q^T Q \begin{pmatrix} R \\ 0 \end{pmatrix} = R^T R, \tag{A-5}$$

which is the required Cholesky factorization of $H^T J^{-1} H$.

If Z is known the above procedure can be applied to the factorization of Z^TJZ. However if Z is not known we first perform the QR factorization of H to determine Z:

$$H = Q_1\begin{pmatrix} R \\ 0 \end{pmatrix} = (Q,\ Q')\begin{pmatrix} R \\ 0 \end{pmatrix} = QR$$

where Q_1 is an MxM orthogonal matrix and R is an NxN upper triangular matrix. Q_1 is partitioned into two matrices Q and Q' such that Q is MxN and Q' is MxM-N. Since $Q'^TQ=0$ (Q_1 is orthogonal) it follows that $Q'^TH=0$ and hence Z=Q'. It should be noted that the general solution to the equation $H^Tx=b$ is given by

$$x = Yb + Zx' \qquad \text{(A-6)}$$

where $Y=QR^{-T}$ and x' is any (M-N) vector. This can be easily seen by noting that Y is the Moore-Penrose generalized inverse of H^T. However we mention that for the solution x to be written in the form of Eq. (A-6) Y need not be the generalized inverse of H^T. All that is required is to find Y such that $H^TY=E_N$. Y^T can be regarded as the left generalized inverse of H.

A general method [13] for finding the matrices Y and Z is to augment the MxN matrix H (M>N) by adding an MxM-N matrix H' such that (H',H) is nonsingular. From the definition of the inverse it follows that

$$(Z,Y)^T = (H',H)^{-1}$$

Let $H'=Q'$. Then by taking $H=QR$ it is easy to check that $Y=QR^{-T}$ and Z=Q'. These are the same expressions as obtained above by the QR method. To obtain some other expressions for Y and Z let us assume that the last N rows of H denoted by H_2 are linearly independent. Then by taking $H'=(E_{M-N},0)^T$ we have

$$\begin{pmatrix} E_{M-N} & H_1 \\ 0 & H_2 \end{pmatrix}^{-1} = \begin{pmatrix} E_{M-N} & -H_1H_2^{-1} \\ 0 & H_2^{-1} \end{pmatrix}$$

which yields $Z^T=(E_{M-N},-H_1H_2^{-1})$ and $Y^T=(0,\ H_2^{-1})$.

Case 2, $K \leq N$

An MxN matrix H, $M \geq N$, and of rank K can be decomposed as

$$H = V_1S_1W_1^T \qquad \text{(A-7)}$$

where V_1 is an MxM orthogonal matrix, W_1 is an NxN orthogonal matrix and S_1 is an MxN matrix given by

$$S_1 = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{(A-8)}$$

with $S = \text{diag}\{s_1,\ldots,s_K\}$. The positive real numbers s_i, i=1,...,K are the singular values of the matrix H. They are often ordered so that

$s_1 \geq s_2 \cdots \geq s_K > 0$. By partitioning V_1 and W_1 so that $V_1=(V,V')$ and $W_1=(W,W')$ and substituting into Eq. (A-7) we obtain

$$H = V\ S\ W^T \tag{A-9}$$

where $V^TV=E_K$ and $W^TW = E_K$. The factorisation (A-7) or (A-9) is called the singular value decomposition [17]. The general solution to the equation $H^Tx=b$, if exists, is given by

$$x = Yb + Zx' \tag{A-10}$$

where $Y = V\ S^{-1}\ W^T$ and $Z = V'$. It should be noted that Y is the Moore-Penrose generalized inverse of H^T.

The singular value decomposition method can also be used when K=N. However, because of the efficiency of the QR decomposition for the case K=N the singular value decomposition should be used only when K<N or when it cannot be established a priori that K=N.

As in the case when rank of H is N we write $H^TJ^{-1}H$ in the form of Eq. (A-3). However instead of taking the QR factorization of the matrix $D^{-\frac{1}{2}}L^{-1}H$ we write the singular value decomposition:

$$D^{-\frac{1}{2}}L^{-1}H = V\ S\ W^T$$

With this value, the matrix $H^T J^{-1} H$ can be written as

$$H^TJ^{-1}H = W\ S\ V^TVSW^T$$

$$= W\ S^2\ W^T \tag{A-11}$$

and a solution to the matrix equation $H^T J^{-1} Hx = u$ is

$$x = W\ S^{-2}\ W^T u \tag{A-12}$$

It is of course assumed that the constraint equations are consistent so that a solution to the matrix equation exists.

For the factorization of the matrix $Z^T JZ$ we first determine Z of full rank M-K by the singular value decomposition and then write the Cholesky factorization of Z^TJZ by the QR method.

REPORT DOCUMENTATION PAGE/PAGE DE DOCUMENTATION DE RAPPORT

REPORT/RAPPORT 1a DME-3	REPORT/RAPPORT 1b NRC No. 22797
REPORT SECURITY CLASSIFICATION CLASSIFICATION DE SÉCURITÉ DE RAPPORT 2 UNCLASSIFIED	DISTRIBUTION/DIFFUSION 3 UNLIMITED

TITLE/SUBTITLE/TITRE/SOUS-TITRE (CLASSIFICATION)

4 Equations of Motion for Rigid Multibody Systems

AUTHOR(S)/AUTEUR(S)

5 Mufti, I.H.

SERIES/SÉRIE

6 Division of Mechanical Engineering Report

CORPORATE AUTHOR/PERFORMING AGENCY/AUTEUR D'ENTREPRISE/AGENCE D'EXÉCUTION

7 National Research Council, Division of Mechanical Engineering — Systems Laboratory

SPONSORING AGENCY/AGENCE DE SUBVENTION

8

DATE	FILE/DOSSIER	LAB. ORDER / COMMANDE DU LAB.	PAGES	DIAGS	REFS
1983/11			36	5	19
9	10	11	12a	12b	12c

NOTES

13

DESCRIPTORS(KEY WORDS)/MOTS-CLÉS

14
1. Non-linear programming
2. Euler-Lagrange equation

SUMMARY/SOMMAIRE

15 A unified presentation of several computer oriented formulations of the equations of motion for multibody systems is given.

ADDRESS/ADRESSE

16 E.H. Dudgeon, Director
Division of Mechanical Engineering
Montreal Road, Ottawa, Canada
K1A 0R6 (613) 993-2424

POOR

QUALITY

ORIGINAL